G02 GRAVITATIONAL POTENTIAL ENERGY IN GENERAL

SPH4U

EXPECTATIONS

- analyze the factors affecting the motion of isolated celestial objects and calculate the gravitational potential energy for each system
- analyze isolated planetary and satellite motion, and describe the motion in terms of the forms of energy and energy transformations that occur

EQUATIONS

- Gravitational Potential Energy

$$\Delta E_g = \frac{GMm}{r}$$

- Escape Velocity

$$v = \sqrt{\frac{2GM}{r}}$$

GRAVITATIONAL POTENTIAL ENERGY IN GENERAL

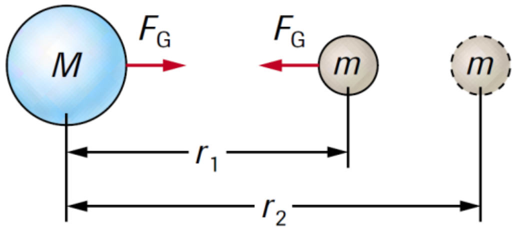
- Recall: for distances that are close together (less than a few hundred km) we use
$$\Delta E_g = mg\Delta y$$
- When we look at greater changes in displacement y, we can no longer approximate g as constant.

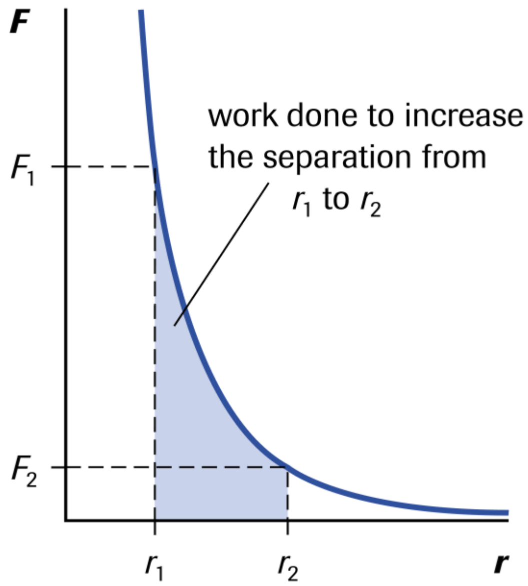
DETERMINING ΔE_g IN GENERAL

- We start with the general gravitational force

$$F_G = \frac{GMm}{r^2}$$

- In order to increase the separation distance between two objects, work needs to be done to overcome the force of attraction
  - Similar to stretching a spring
- Note: this work is equal to the area under the Force-separation graph between r_1 and r_2

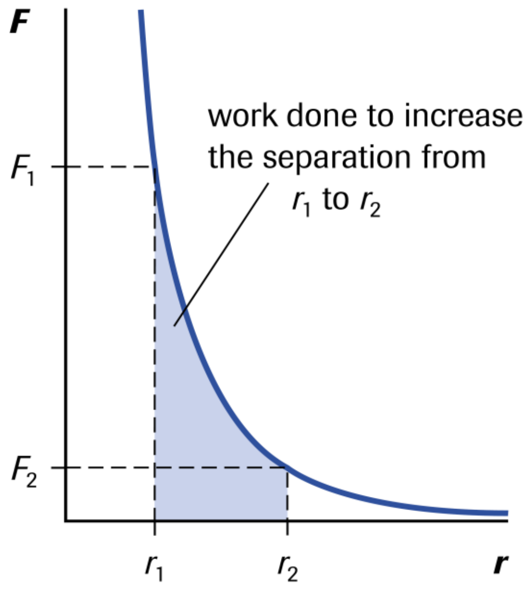




DETERMINING ΔE_g IN GENERAL

- To avoid the use of calculus, we take the geometric average of F_1 and F_2 ($\sqrt{F_1 F_2}$)
- The area under the curve is the work done to increase the separation, but is also the change in gravitational potential energy

$$
\begin{aligned}
\text{area} &= \sqrt{F_1 F_2}(r_2 - r_1) \\
&= \sqrt{\left(\frac{GMm}{r_1^2}\right)\left(\frac{GMm}{r_2^2}\right)}(r_2 - r_1) \\
&= \frac{GMm}{r_1 r_2}(r_2 - r_1) \\
&= \frac{GMm}{r_1} - \frac{GMm}{r_2}
\end{aligned}
$$



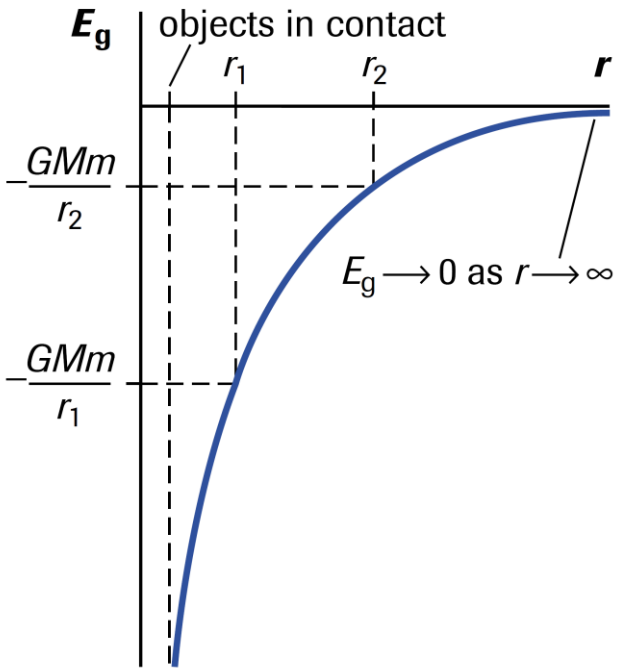
DETERMINING ΔE_g IN GENERAL

- Since the area is the change in gravitational potential energy,

$$\text{area} = \Delta E_g = \frac{GMm}{r_1} - \frac{GMm}{r_2}$$
$$= \left(-\frac{GMm}{r_2}\right) - \left(-\frac{GMm}{r_1}\right)$$
$$= E_{g_2} - E_{g_1}$$

- So, in general,

$$\boldsymbol{E_g = -\frac{GMm}{r}}$$



PROBLEM 1

What is the change in gravitational potential energy of a 64.5-kg astronaut, lifted from Earth's surface into a circular orbit of altitude 4.40×10^2 km?

$G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$	$m = 64.5\ \text{kg}$
$M_E = 5.98 \times 10^{24}\ \text{kg}$	$r_E = 6.38 \times 10^6\ \text{m}$

PROBLEM 1 – SOLUTIONS

$G = 6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2$ $\quad m = 64.5\ \text{kg}$

$M_E = 5.98 \times 10^{24}\ \text{kg}$ $\quad r_E = 6.38 \times 10^6\ \text{m}$

$$\begin{aligned} r_2 &= r_E + 4.40 \times 10^2\ \text{km} \\ &= 6.38 \times 10^6\ \text{m} + 4.40 \times 10^5\ \text{m} \\ r_2 &= 6.82 \times 10^6\ \text{m} \end{aligned}$$

On Earth's surface,

$$\begin{aligned} E_{g1} &= -\frac{GM_Em}{r_E} \\ &= -\frac{(6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(64.5\ \text{kg})}{6.38 \times 10^6\ \text{m}} \\ E_{g1} &= -4.03 \times 10^9\ \text{J} \end{aligned}$$

In orbit,

$$\begin{aligned} E_{g2} &= -\frac{GM_Em}{r_2} \\ &= -\frac{(6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(64.5\ \text{kg})}{6.82 \times 10^6\ \text{m}} \\ E_{g2} &= -3.77 \times 10^9\ \text{J} \end{aligned}$$

$$\begin{aligned} \Delta E_g &= E_{g2} - E_{g1} \\ &= (-3.77 \times 10^9\ \text{J}) - (-4.03 \times 10^9\ \text{J}) \\ \Delta E_g &= 2.6 \times 10^8\ \text{J} \end{aligned}$$

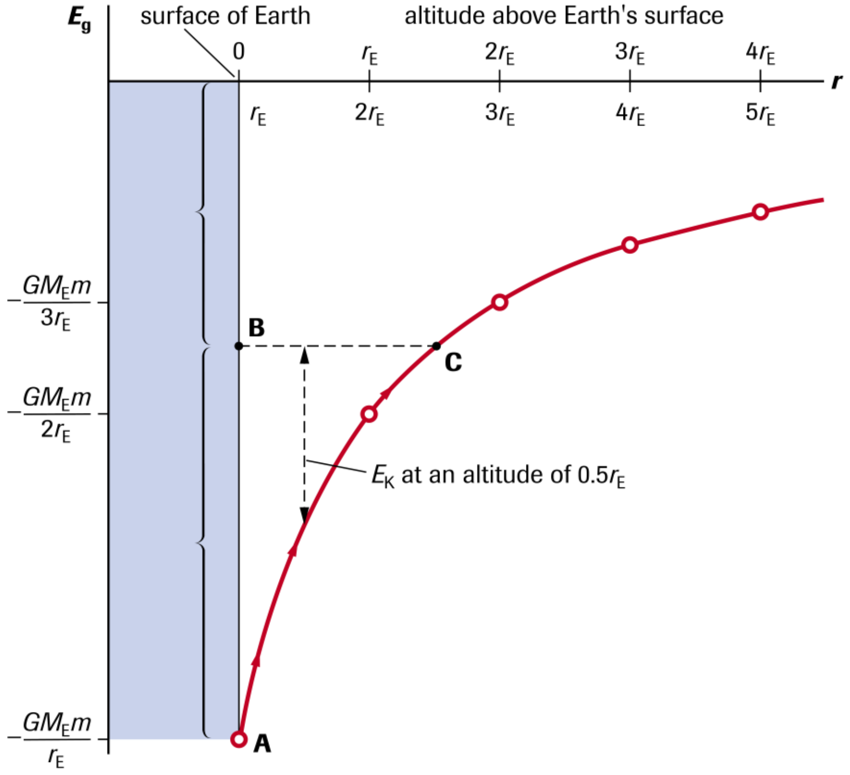
The change in gravitational potential energy is 2.6×10^8 J.

ESCAPE FROM A GRAVITATIONAL FIELD

- Recall: the gravitational potential energy of any two masses separated by a distance r is

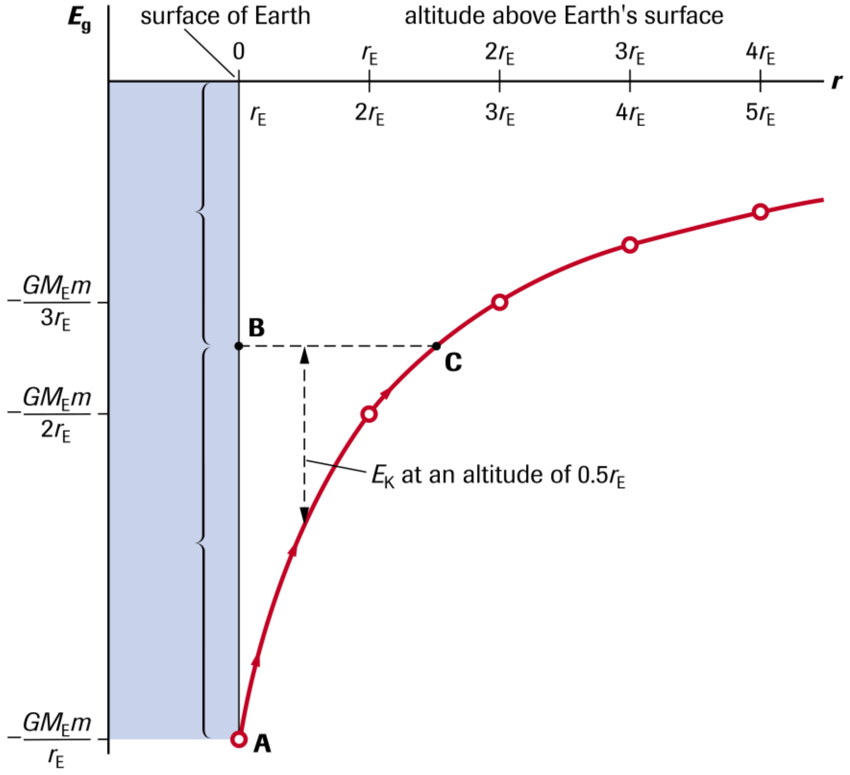
$$E_g = -\frac{GMm}{r}$$

- The negative gives the shape of a potential well, the graph of E_g vs r
- Recall: $E_T = E_K + E_g$



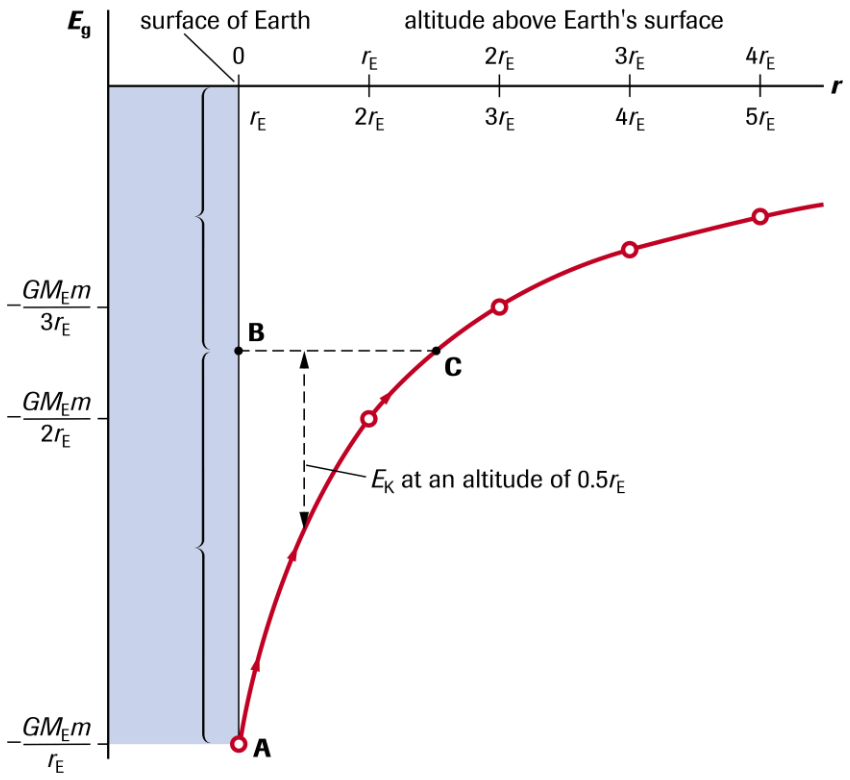
ESCAPE FROM A GRAVITATIONAL FIELD

- Consider a rocket at rest on the surface of a planet (point A)
  - $E_K = 0$
  - $E_g = -\frac{GMm}{r}$ (minimum value)
- Launching the rocket will provide an initial kinetic energy (line AB)
  - E_g will increase along the curve to C
  - E_K will decrease to 0 at C
  - $E_T = E_K + E_g =$ constant (line BC)
- Note: once $E_K = 0$, the rocket falls back to the surface, converting E_g back to E_K



ESCAPE FROM A GRAVITATIONAL FIELD

- To escape a potential well, we need to reach a separation large enough to have a gravitational potential energy of zero
  - $r \to \infty, E_g \to 0$
- If we have a minimum kinetic energy to reach this distance, we will be left with zero kinetic energy at this distance
  - $r \to \infty, E_K \to 0$



ESCAPE FROM A GRAVITATIONAL FIELD

- **Escape Speed:** the minimum speed needed to project a mass m from the surface of mass M to just escape the gravitational force of M

$$E_T = E_K + E_g = 0$$

$$E_K = -E_g$$

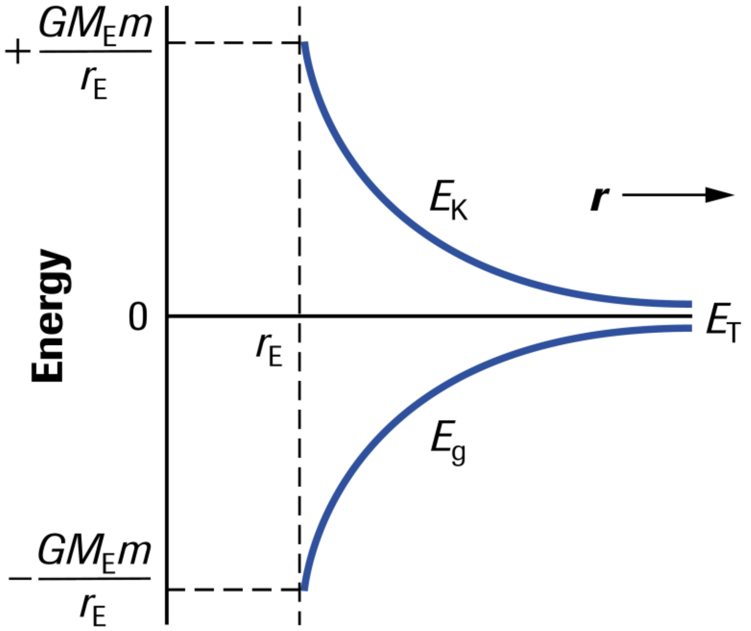
$$\frac{1}{2}mv^2 = -\left(-\frac{GMm}{r}\right)$$

$$\boldsymbol{v = \sqrt{\frac{2GM}{r}}}$$

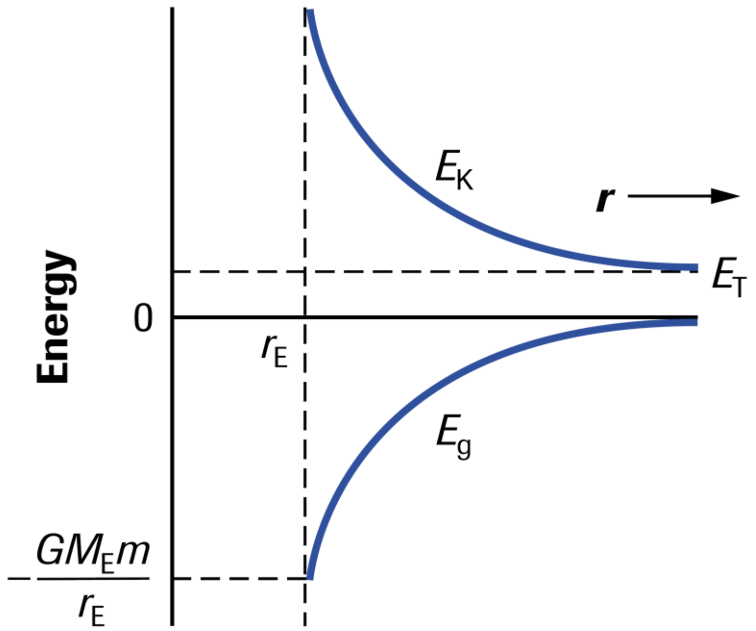
ESCAPE FROM A GRAVITATIONAL FIELD

- **Escape Energy:** the minimum kinetic energy needed to project a mass m from the surface of mass M to just escape the gravitational force of M
  - When $E_T = E_K + E_g = 0$, then $E_K = -E_g$
  - When $E_T > 0$, then $E_K > -E_g$ and m will have some E_K left at an infinite separation distance
  - When $E_T < 0$, then $E_K < -E_g$ and m will not escape, but will stop at a finite distance before falling back down to the surface

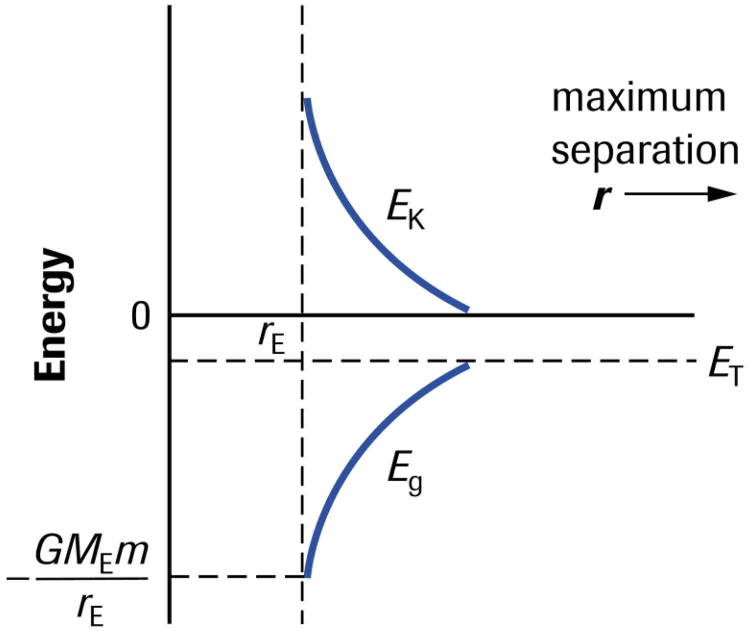
Case 1: $E_T = 0$, object just escapes



Case 2: $E_T > 0$, object escapes with a speed > 0 as $r \to \infty$



Case 3: $E_T < 0$, object is bound to Earth

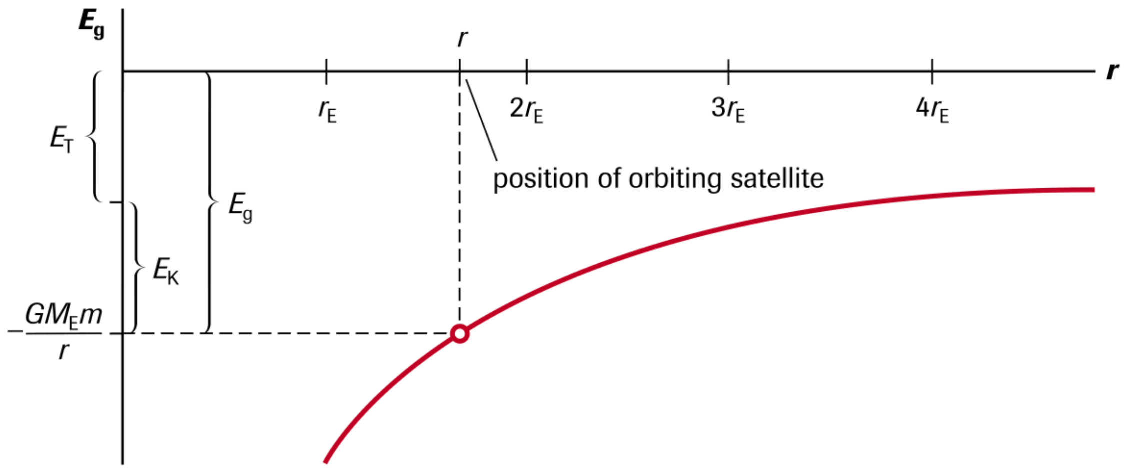


ESCAPE FROM A GRAVITATIONAL FIELD

- **Binding Energy:** the amount of additional kinetic energy needed by a mass m to just escape from a mass M
  - At the surface:

$$E_T = E_K + E_g$$
$$= 0 + \left(-\frac{GMm}{r}\right)$$
$$= -\frac{GMm}{r}$$

  - The binding energy must be $\frac{GMm}{r}$ to escape the surface



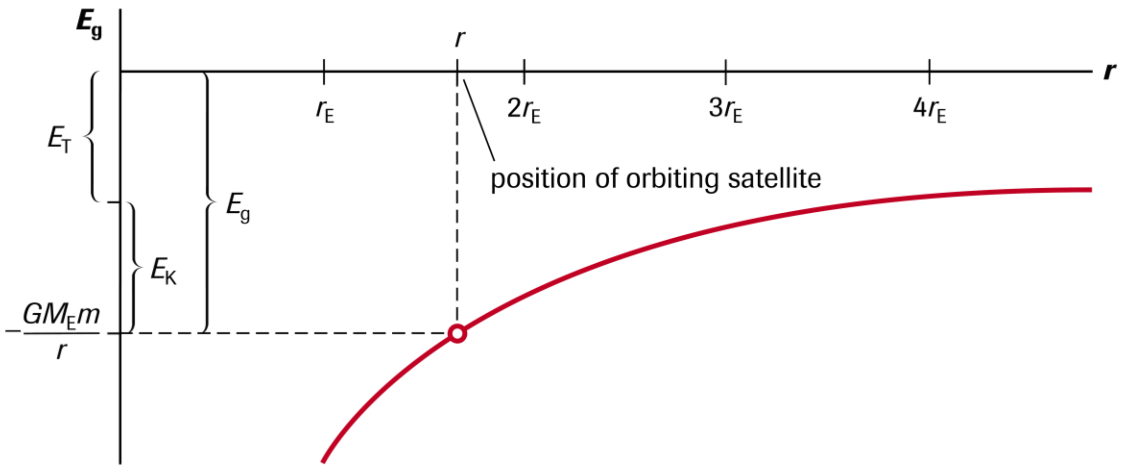
ESCAPE FROM A GRAVITATIONAL FIELD

- **Binding Energy:** the amount of additional kinetic energy needed by a mass m to just escape from a mass M
  - In orbit:

$$\Sigma F = F_G$$

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$mv^2 = \frac{GMm}{r}$$



ESCAPE FROM A GRAVITATIONAL FIELD

- **Binding Energy:** the amount of additional kinetic energy needed by a mass m to just escape from a mass M
  - In orbit:

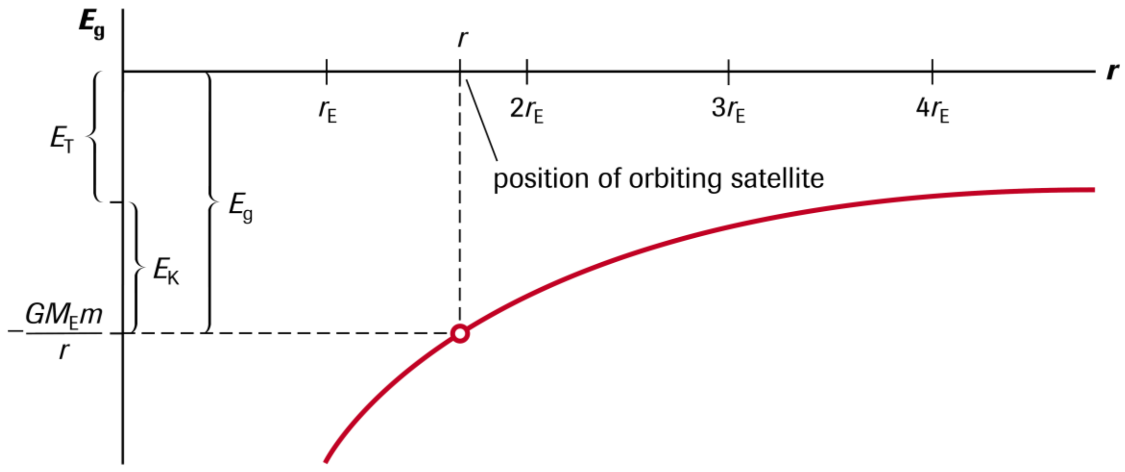
$$E_T = E_K + E_g$$

$$E_T = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$= \frac{1}{2}\left(\frac{GMm}{r}\right) - \frac{GMm}{r}$$

$$= -\frac{1}{2}\frac{GMm}{r}$$

$$= \frac{1}{2}E_g$$



PROBLEM 2

A 5.00×10^2-kg communications satellite is to be placed into a circular geosynchronous orbit around Earth. (A geosynchronous satellite remains in the same relative position above Earth because it has a period of 24.0 h, the same as that of Earth's rotation on its axis.)

(a) What is the radius of the satellite's orbit?

(b) What is the gravitational potential energy of the satellite when it is attached to its launch rocket, at rest on Earth's surface?

(c) What is the total energy of the satellite when it is in geosynchronous orbit?

(d) How much work must the launch rocket do on the satellite to place it into orbit?

(e) Once in orbit, how much additional energy would the satellite require to escape from Earth's potential well?

PROBLEM 2 – SOLUTIONS

(a) $G = 6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2$

$T = 24.0\ \text{h} = 8.64 \times 10^4\ \text{s}$

$M_E = 5.98 \times 10^{24}\ \text{kg}$

As for any satellite:

$$\sum F = F_G$$

$$\frac{4\pi^2 mr}{T^2} = \frac{GM_E m}{r^2}$$

$$r = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}}$$

$$= \sqrt[3]{\frac{(6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(8.64 \times 10^4\ \text{s})^2}{4\pi^2}}$$

$$r = 4.22 \times 10^7\ \text{m}$$

The radius of the satellite's orbit is 4.22×10^7 m. This radius represents an altitude of 3.58×10^4 km above Earth's surface.

PROBLEM 2 – SOLUTIONS CONT.

(b) $r_E = 6.38 \times 10^6$ m

$m = 5.00 \times 10^2$ kg

At the surface of Earth,

$$E_g = -\frac{GM_E m}{r_E}$$

$$= -\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(5.00 \times 10^2\ \text{kg})}{6.38 \times 10^6\ \text{m}}$$

$$E_g = -3.13 \times 10^{10}\ \text{J}$$

The gravitational potential energy of the satellite when it is attached to its launch rocket at rest on Earth's surface is -3.13×10^{10} J.

PROBLEM 2 – SOLUTIONS CONT.

(c) $r = 4.22 \times 10^7$ m

The total energy of a satellite in circular orbit, bound to Earth, is given by:

$$
\begin{aligned}
E_T &= E_K + E_g \\
&= \frac{1}{2}mv^2 - \frac{GM_E m}{r} \\
&= -\frac{1}{2}\frac{GM_E m}{r} \quad \text{(based on the theory related to } \textbf{Figure 5}\text{)} \\
&= -\frac{1}{2}\frac{(6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(5.00 \times 10^2\ \text{kg})}{4.22 \times 10^7\ \text{m}} \\
E_T &= -2.36 \times 10^9\ \text{J}
\end{aligned}
$$

The total energy of the satellite when in geosynchronous orbit is -2.36×10^9 J.

PROBLEM 2 – SOLUTIONS CONT.

(d)
$$W = \Delta E = E_T\text{ (in orbit)} - E_T\text{ (on Earth)}$$
$$= -2.36 \times 10^9\text{ J} - (-3.13 \times 10^{10}\text{ J})$$
$$W = 2.89 \times 10^{10}\text{ J}$$

The launch rocket must do 2.89×10^{10} J of work on the satellite to place it into orbit.

(e) To escape Earth's potential well, the total energy of the satellite must be zero or greater. In orbit, $E_T = -2.36 \times 10^9$ J. Therefore, to escape Earth's potential well, the satellite must acquire at least 2.36×10^9 J of additional energy.

SUMMARY

- The gravitational potential energy of a system of two (spherical) masses is directly proportional to the product of their masses, and inversely proportional to the distance between their centres.
- A gravitational potential energy of zero is assigned to an isolated system of two masses that are so far apart (i.e., their separation is approaching infinity) that the force of gravity between them has dropped to zero.
- The change in gravitational potential energy very close to Earth's surface is a special case of gravitational potential energy in general.
- Escape speed is the minimum speed needed to project a mass m from the surface of mass M to just escape the gravitational force of M.
- Escape energy is the minimum kinetic energy needed to project a mass m from the surface of mass M to just escape the gravitational force of M.
- Binding energy is the amount of additional kinetic energy needed by a mass m to just escape from a mass M.

PRACTICE

Readings

- Section 6.3 (pg 285)

Questions

- pg 294 #1,2,4,5,6